Eulerian spectral closures for isotropic turbulence using a time-ordered fluctuation-dissipation relation

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(Received 17 February 2005; published 19 July 2005)

Procedures for time-ordering the covariance function, as given in a previous paper [K. Kiyani and W. D. McComb, Phys. Rev. E **70**, 066303 (2004)], are extended and used to show that the response function associated at second order with the Kraichnan-Wyld perturbation series can be determined by a local (in wave number) energy balance. These time-ordering procedures also allow the two-time formulation to be reduced to time-independent form by means of exponential approximations and it is verified that the response equation does not have an infrared divergence at infinite Reynolds number. Last, single-time Markovianized closure equations (stated in our previous paper) are derived and shown to be compatible with the Kolmogorov distribution without the need to introduce an *ad hoc* constant.

DOI: 10.1103/PhysRevE.72.016309 PACS number(s): 47.27.Ak, 47.27.Eq, 47.27.Gs, 05.20.−y

I. INTRODUCTION

In a previous paper [1], the Kraichnan-Wyld perturbation expansion [2,3] was used to justify the introduction of a renormalized response function connecting two-point covariances at different times. The resulting relationship was specialized by suitable choice of initial conditions to the form of a fluctuation-dissipation relation (FDR). This was further developed to reconcile the time symmetry of the covariance with the causality of the response function by the introduction of time ordering along with a counterterm. We pointed out that this formulation provides a solution to an old problem in turbulence theory: that of representing the time dependence of the covariance and response by exponential forms [4,5]. We showed that the derivative (with respect to difference time) of the covariance with this time ordering now vanishes at the origin. This allows one to study the relationships between two-time spectral closures and time-independent theories such as the Fokker-Planck theory of Edwards [6] or the more recent renormalization group approaches. We also showed that the renormalized response function is transitive with respect to intermediate times and reported a different Langevin-type equation for turbulence.

In this paper we interpret the second-order response function as a mean-field propagator and show that in addition to propagating two-time covariances it also links single-time covariances. We then make use of its newly established properties to rederive the local-energy transfer (LET) response equation [7] and show that it now contains a counterterm which removes the singularity of previous propagator equations at $t=t'$. We also introduce a partial-propagator representation and hence reformulate the LET statistical equations. Furthermore we specialize the two-time equations to time-independent form by introducing exponential time dependences and show that the closure is well behaved in the limit of infinite Reynolds number. Last, by Markovianizing time-history integrals, we end up with a Langevin-type theory which is compatible with the Kolmogorov spectrum without the need to introduce *ad hoc* constants as in the case of the EDQNM model [8] for example.

We begin by reviewing the subject of turbulence closures and then go on to consider various aspects of applying the FDR to nonequilibrium, macroscopic problems such as fluid turbulence. We begin by stating the basic equations.

A. The basic equations

Following standard practice in this topic [9], we consider the solenoidal Navier-Stokes equation (NSE) in wave number (k) space, as follows:

$$\left(\frac{\partial}{\partial t}+\nu_0 k^2\right)u_\alpha(\mathbf{k},t)=M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t), \tag{1}$$

while the continuity equation for incompressible fluids is

$$k_\alpha u_\alpha(\mathbf{k},t)=0. \tag{2}$$

The inertial transfer operator $M_{\alpha\beta\gamma}(\mathbf{k})$ is given by

$$M_{\alpha\beta\gamma}(\mathbf{k})=(2i)^{-1}[k_\beta P_{\alpha\gamma}(\mathbf{k})+k_\gamma P_{\alpha\beta}(\mathbf{k})], \tag{3}$$

while the projector $P_{\alpha\beta}(\mathbf{k})$ is expressed in terms of the Kronecker δ as

$$P_{\alpha\beta}(\mathbf{k})=\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{|\mathbf{k}|^2}. \tag{4}$$

In order to introduce a statistical treatment, we shall denote the operation of performing an ensemble average by angle brackets, thus $\langle\cdots\rangle$, and restrict our attention to isotropic, homogeneous turbulence, with energy dissipation rate ε and zero mean velocity. As a result of this restriction, the covariance of the fluctuating velocity field takes the form

$$\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{k}',t')\rangle=C(k;t,t')P_{\alpha\beta}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}'), \tag{5}$$

where $\alpha,\beta=1$, 2 or 3. The corresponding single-time quantity may be written as

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$$C(k;t,t) = C(k,t), \tag{6}$$

where the single-time one-point covariance $C(k,t)$ may be interpreted as a spectral density and is related to the energy spectrum by

$$E(k,t) = 4\pi k^2 C(k,t). \tag{7}$$

Using Eq. (1), we can also obtain an equation describing the energy balance between spatial modes. To do this, we first multiply each term in Eq. (1) by $u_\sigma(-\mathbf{k},t)$. Then we form a second equation from (1) for $u_\sigma(-\mathbf{k},t)$, multiply this by $u_\alpha(\mathbf{k},t)$, add the two resulting equations together, integrate over $\mathbf{k}'$, and average the final expression. This leaves us with

$$\begin{aligned}\left(\frac{\partial}{\partial t} + 2\nu_0 k^2\right) & P_{\alpha\sigma}(\mathbf{k}) C(k,t) \\ &= M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j\, C_{\sigma\beta\gamma}(-\mathbf{k},\mathbf{j},\mathbf{k}-\mathbf{j};t) \\ &\quad - M_{\sigma\beta\gamma}(\mathbf{k}) \int d^3j\, C_{\sigma\beta\gamma}(\mathbf{k},\mathbf{j},-\mathbf{k}-\mathbf{j};t),\end{aligned} \tag{8}$$

where

$$C_{\alpha\beta\gamma}(\mathbf{k},\mathbf{j},-\mathbf{k}-\mathbf{j};t) = \langle u_\alpha(\mathbf{k},t) u_\beta(\mathbf{j},t) u_\gamma(-\mathbf{k}-\mathbf{j},t)\rangle, \tag{9}$$

and where we have also used the property

$$M_{\alpha\beta\gamma}(-\mathbf{k}) = -M_{\alpha\beta\gamma}(\mathbf{k}). \tag{10}$$

If we then take the trace of Eq. (8) by setting $\sigma=\alpha$ and summing over α (noting that $\mathrm{Tr}\, P_{\alpha\beta}=2$), and multiply each term in Eq. (8) by $2\pi k^2$, we obtain

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) E(k,t) = T(k,t), \tag{11}$$

where

$$\begin{aligned} T(k,t) = 2\pi k^2 M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \\ \times\{C_{\alpha\beta\gamma}(-\mathbf{k},\mathbf{j},\mathbf{k}-\mathbf{j},t) - C_{\alpha\beta\gamma}(\mathbf{k},\mathbf{j},-\mathbf{k}-\mathbf{j},t)\}.\end{aligned} \tag{12}$$

Evidently, in order to solve for the energy spectrum (or, second-order moment) we need to know the third-order moment. Hence we are faced with a hierarchy of statistical equations to be solved; and this is, of course, the notorious closure problem.

B. Eulerian statistical closures for isotropic turbulence

In order to study isotropic turbulence, we have to add a noise term or stirring force to the right hand side of the NSE, as given by Eq. (1). Denoting this term by $f_\alpha(\mathbf{k},t)$, we specify it in terms of its distribution, which we take to be Gaussian, and its covariance, which we take to be of the form

$$\langle f_\alpha(\mathbf{k},t) f_\beta(\mathbf{k}',t')\rangle = W(k)(2\pi)^3 P_{\alpha\beta}(\mathbf{k})\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(t+t'). \tag{13}$$

We note that $W(k)$ is a measure of the rate at which the stirring forces do work on the fluid and for stationarity must satisfy the condition

$$\int_0^\infty 4\pi k^2 W(k) dk = \varepsilon = \int_0^\infty 2\nu k^2 E(k) dk. \tag{14}$$

The perturbative treatment of the equations of motion is based on an expansion about a Gaussian zero-order velocity obtained by solving the NSE with the nonlinear term set to zero. The resulting expansion shows clearly [3] the effect of nonlinear mixing such that any correction to the zero-order field must have a non-Gaussian distribution, which indeed is implied by the existence of the third-order moment (and the existence of intermodal energy transfer). Renormalization of the perturbation expansion corresponds to either partial summation or term-by-term reversion: for details reference should be made to the paper by Wyld [3] and the books by McComb [9] and Leslie [5]. Our present interest is restricted to the second-order equation for the velocity covariance, which is obtained by this procedure, thus:

$$\begin{aligned}\left[\frac{\partial}{\partial t} + \nu k^2\right] & C(k;t,t') \\ &= \int d^3j L(\mathbf{k},\mathbf{j}) \left[\int_0^{t'} ds\, R(k;t',s) C(j;t,s) C(|\mathbf{k}-\mathbf{j}|;t,s) \right.\\ &\quad \left. - \int_0^t ds\, R(j;t,s) C(k;s,t') C(|\mathbf{k}-\mathbf{j}|;t,s)\right],\end{aligned} \tag{15}$$

and on the time diagonal

$$\begin{aligned}\left(\frac{\partial}{\partial t} + 2\nu k^2\right) & C(k,t) \\ &= 2\int d^3j\, L(\mathbf{k},\mathbf{j}) \times \int_0^t ds\, R(k;t,s) R(j;t,s) R(|\mathbf{k}-\mathbf{j}|;t,s) \\ &\quad \times [C(j,s)C(|\mathbf{k}-\mathbf{j}|,s) - C(k,s)C(|\mathbf{k}-\mathbf{j}|,s)],\end{aligned} \tag{16}$$

where the coefficient $L(\mathbf{k},\mathbf{j})$ is given by

$$L(\mathbf{k},\mathbf{j}) = -2M_{\alpha\beta\gamma}(\mathbf{k}) M_{\beta\alpha\delta}(\mathbf{j}) P_{\gamma\delta}(\mathbf{k}-\mathbf{j}). \tag{17}$$

This may be evaluated in terms of the scalar magnitudes k,j and $\mu=\cos\theta$, where θ is the angle between the two wave vectors $\mathbf{k}$ and $\mathbf{j}$; thus

$$L(\mathbf{k},\mathbf{j}) = \frac{[\mu(k^2+j^2) - kj(1+2\mu^2)](\mu^2-1)kj}{k^2+j^2-2kj\mu}. \tag{18}$$

It should also be noted that the coefficient $L(\mathbf{k},\mathbf{j})$ is symmetric under interchange of the two wave vectors: we shall use this fact presently to demonstrate conservation of energy.

At this stage we should note that for this to be a closed set of equations for the covariance C, one has to have an additional equation to determine the response function R. Equation (15) was originally derived by Kraichnan. This closure was completed by an equation for the response function

$R(k;t,t')$, and is known as the *direct interaction approximation* (DIA). The basic ansatz of DIA is that there exists a response function such that

$$\delta u_\alpha(\mathbf{k},t) = \int_{-\infty}^{t} \hat{R}_{\alpha\beta}(\mathbf{k};t,t')\,\delta f_\beta(\mathbf{k},t')dt', \qquad (19)$$

and that this *infinitesimal response* function can be renormalized. The resulting response equation is

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] R(k;t,t') + \int d^3 j\, L(\mathbf{k},\mathbf{j}) \times \int_{t'}^{t} dt'' R(k;t'',t') R(j;t,t'') C(|\mathbf{k}-\mathbf{j}|;t,t'') = \delta(t-t'). \qquad (20)$$

Later Edwards derived a time-independent covariance equation by the self-consistent introduction of a generalized Fokker-Planck equation as an approximation to the (rigorous) Liouville equation. We shall refer to this theory as the EFP theory, and this along with the more general *self-consistent field* (SCF) theory of Herring [10] and the DIA make up our trio of pioneering spectral closures. Further discussion can be found in the books [5,9]. In the literature, much attention has been given to the fact that, although these theories have many satisfactory features, they are all incompatible with the Kolmogorov 1941 (K41) power law for the energy spectrum $E(k)$ [11]. However, in the present paper we shall concentrate on only a few key points. The first of these is that the covariance equation of the EFP theory can be shown to be equivalent, for the stationary case, to the second-order truncation of the Kraichnan-Wyld perturbation theory, if we assume exponential time dependences. That is, the EFP covariance equation can be obtained by substituting into Eq. (15) the following assumed time dependences:

$$C(k,t-t') = C(k)\exp\{-\omega(k)|t-t'|\}, \qquad (21)$$

and

$$R(k,t-t') = \exp\{-\omega(k)(t-t')\} \text{ for } t \geqslant t'$$
$$=0 \text{ for } t < t'. \qquad (22)$$

Then, integrating the right hand side of Eq. (15) over intermediate times, one obtains (with some rearrangement)

$$W(k) - 2\nu k^2 C(k) = \int d^3 j\, L(k,j) \frac{C(|\mathbf{k}-\mathbf{j}|)[C(k)-C(j)]}{\omega(k)+\omega(j)+\omega(|\mathbf{k}-\mathbf{j}|)}, \qquad (23)$$

where we have added the term $W(k)$ to the energy balance in order to sustain the turbulence against viscous dissipation. Equation (23) is just the form originally derived by Edwards [6].

This simple form is helpful in understanding certain properties, such as the conservation of energy by the nonlinear term and the behaviour of the system in the limit of infinite Reynolds number. For instance, integrating both sides of Eq. (23) with respect to **k** and invoking Eq. (14) leads to

$$\varepsilon - \varepsilon = 0, \qquad (24)$$

where the vanishing of the right hand side results from the antisymmetry of the integrand under interchange of k and j. This result helps us to interpret the EFP response function or eddy decay rate $R(k)$ which takes the form [6]

$$R(k) = \int d^3 j\, L(k,j) \frac{C(|\mathbf{k}-\mathbf{j}|)}{\omega(k)+\omega(j)+\omega(|\mathbf{k}-\mathbf{j}|)}. \qquad (25)$$

At the time this was interpreted as allowing one to write the energy balance equation as

$$W(k) - 2\nu k^2 C(k) = R(k)C(k) - \int d^3 j\, L(k,j) \frac{C(|\mathbf{k}-\mathbf{j}|)C(j)}{\omega(k)+\omega(j)+\omega(|\mathbf{k}-\mathbf{j}|)}, \qquad (26)$$

or the eddy decay rate represents the loss of energy from mode k due to energy transfers to all other modes. The situation is more complicated for the DIA, but the analogous comment has been made by Kraichnan [2] that the energy loss from mode k is directly proportional to the excitation of that mode, viz., $C(k;t,t')$.

Later, it was pointed out that an *ad hoc* modification could be made to the EFP theory by noting that the entire energy transfer term [i.e., the right hand side of Eq. (23)] acts as an energy loss in some regions of wave number space whereas in others it behaves as an input. This led to a definition of the response which was compatible with the Kolmogorov spectrum [12,13] and this was subsequently generalized to the two-time *local energy transfer* or LET theory [14].

We have restricted our attention to Eulerian closures in this section but we should mention the Lagrangian closures which also claim compatibility with K41. In the interest of completeness we shall say something about these when we give an overview of the subject in the Conclusion.

C. Fluctuation-dissipation relations

It is well known that the response of microscopic systems in thermal equilibrium to small perturbations is fully determined by the covariance of the system fluctuations about equilibrium. In our present notation, the relationship may be written as

$$C(k;t,t') = R(k;t,t')C(k;t',t'), \qquad (27)$$

which is the fluctuation-dissipation relation. This result was extended by Kraichnan to nonlinear dynamical systems in thermal equilibrium [15,16] and by Leith [17] to inviscid two-dimensional chaotic flow. Also, Deker and Haake [18] give several examples of classical processes for which a FDR will hold and these include (of particular relevance to the present discussion) forced viscous flows where the stationary probability distribution is Gaussian. In realistic cases, such flows will have a non-Gaussian distribution due to nonlinear mode coupling. However, one case of interest arises in a pioneering application of renormalization group methods to stirred fluid motion [19], where a fluctuation-dissipation relationship is found to hold in the limit $k \to 0$. That is, the long-wavelength behavior at lowest nontrivial order of perturbation theory.

The paper by Leith is particularly interesting. While recognizing that the FDR cannot apply exactly to real fluid turbulence, it puts forward rather convincing heuristic arguments for believing that it could be a reasonable approximation. Also, it cites the investigation of Herring and Kraichnan [20] in support of this view. Here the nonstationary generalization of the SCF [21], which differs only from the DIA in the use of the FDR, gives very similar results to it. We shall discuss this use of the FDR in more detail later, when we consider its role in the LET theory.

Leith's optimistic view not only inspired successful practical applications of the FDR to study climate sensitivity [22] and viscosity renormalization [23], but was also seen as seminal in stimulating an important series of papers which examined the applicability of the FDR from the point of view of dynamical systems theory [24–27]. The overall conclusion of these papers can be stated as follows.

(1) A general relationship exists for the response of a chaotic system in terms of its stationary probability distribution provided that the system is dynamically mixing.

(2) If the stationary probability is Gaussian in form, then the relationship reduces to the FDR as given by Eq. (27).

Of course in real fluid turbulence the probability distribution is not Gaussian, nor is it known exactly. However, as we have shown in [1], the FDR can be derived for turbulence to second order in renormalized perturbation theory and hence, if used appropriately, is consistent with a closure approximation of that order. We shall return to this point later.

D. The time-ordered FDR

In [1] we postulated that in the context of the Kraichnan-Wyld perturbation theory we may rewrite the existing relationship between the zero-order covariance and zero-order response in a *renormalized* form as

$$C_{\alpha\sigma}(\mathbf{k};t,t') = \theta(t-s)R_{\alpha\epsilon}(\mathbf{k};t,s)C_{\epsilon\sigma}(\mathbf{k};s,t'), \tag{28}$$

or in its isotropic version as

$$C(k;t,t') = \theta(t-s)R(k;t,s)C(k;s,t'), \tag{29}$$

where the Heaviside unit step function $\theta(t-s)$ explicitly states the causality condition. As yet we have taken no decision about the ordering of the two times t and t', and thus the symmetry under interchange of t and t' is untested in Eq. (29).

If we explicitly state the time ordering as $t > t'$ say, then this is equivalent to applying $\theta(t-t')$ to both sides of Eq. (29):

$$\theta(t-t')C(k;t,t') = \theta(t-t')\theta(t-s)R(k;t,s)C(k;s,t'), \tag{30}$$

and this is the beginning of the LET theory. In it, we have postulated the existence of a renormalized propagator and have made use of the Heaviside unit step function to make the time-ordering manifest.

The generalized fluctuation dissipation relationship is obtained by setting $s=t'$ in Eq. (30) to get

$$\theta(t-t')C(k;t,t') = \theta(t-t')R(k;t,t')C(k;t',t'), \tag{31}$$

where the time ordering is set by the requirement $s=t'$.

In [1] we introduced a representation of the covariance which preserves its symmetry under interchange of time arguments; thus

$$C(k;t,t') = \theta(t-t')C(k;t,t') + \theta(t'-t)C(k;t,t') - \delta_{t,t'}C(k;t,t'). \tag{32}$$

We can easily show that this representation does what it is supposed to do by looking in turn at the separate cases $t<t', t>t'$, and $t=t'$, and this is left for the reader.

Now, using Eq. (30) to expand the right hand side of Eq. (32) we obtain

$$C(k;t,t') = \theta(t-t')\theta(t-s)R(k;t,s)C(k;s,t') + \theta(t'-t)\theta(t'-p)R(k;t',p)C(k;p,t) - \delta_{t,t'}C(k;t,t'). \tag{33}$$

Or, this result may be written more like the FDR by instead using Eq. (31) to construct it,

$$C(k;t,t') = \theta(t-t')R(k;t,t')C(k;t',t') + \theta(t'-t)R(k;t',t)C(k;t,t) - \delta_{t,t'}C(k;t,t'). \tag{34}$$

The symmetry of both these covariances Eqs. (33) and (34) can be broken by applying a unit step function to both sides. This will yield either (30) or (31), depending on which time ordering we choose.

II. THE PROPERTIES OF THE MEAN-FIELD PROPAGATOR

In this section we begin by reviewing the introduction of a velocity propagator, as in the original formulation of the LET theory [14] and note that in this context the propagator introduced in [1] is a mean-field propagator.

A. The velocity field propagator

From the exact solution of the solenoidal NSE (see [9]), we have

$$u_\alpha(\mathbf{k},t) = \hat{R}^{(0)}_{\alpha\sigma}(\mathbf{k};t,s)u_\sigma(\mathbf{k},s) + \left[\lambda\int_s^t dt''\hat{R}^{(0)}_{\alpha\sigma}(\mathbf{k};t,t'') \times \int d^3j\, M_{\sigma\beta\gamma}(\mathbf{k})u_\beta(\mathbf{j},t'')u_\gamma(\mathbf{k}-\mathbf{j},t'')\right], \tag{35}$$

where $\hat{R}^{(0)}_{\alpha\sigma}$ is the "viscous" or zero-order response tensor and the caret is used to emphasize that this is the "response" associated with the instantaneous velocity field.

Expanding $u_\alpha(\mathbf{k},t)$ in a perturbation series around a Gaussian solution and equating zero-order terms we can say that the equality

$$u^{(0)}_\alpha(\mathbf{k},t) = \hat{R}^{(0)}_{\alpha\sigma}(\mathbf{k};t,s)u^{(0)}_\alpha(\mathbf{k},s) \tag{36}$$

illustrates the propagatorlike nature of $\hat{R}^{(0)}_{\alpha\sigma}(\mathbf{k};t,s)$. Then from looking at the form of Eq. (35), we can *postulate* the existence of a renormalized propagator such that we obtain a renormalized version of Eq. (36):

$$u_\alpha(\mathbf{k},t) = \hat{R}_{\alpha\sigma}(\mathbf{k};t,s)u_\alpha(\mathbf{k},s). \tag{37}$$

Multiply Eq. (37) by $u_\beta(-\mathbf{k},t')$

$$u_\alpha(\mathbf{k},t)u_\beta(-\mathbf{k},t') = \hat{R}_{\alpha\sigma}(\mathbf{k};t,s)u_\sigma(\mathbf{k},s)u_\beta(-\mathbf{k},t') \tag{38}$$

and average this equation to obtain

$$C_{\alpha\beta}(\mathbf{k};t,t') = R_{\alpha\sigma}(\mathbf{k};t,s)C_{\sigma\beta}(\mathbf{k};s,t'), \tag{39}$$

where the propagator is statistically independent of the velocity field and we have used the mean-field approximation

$$\langle \hat{R}_{\alpha\sigma}(\mathbf{k};t,s)\rangle = R_{\alpha\sigma}(\mathbf{k};t,s). \tag{40}$$

As usual, Eq. (39) can be turned into a simpler scalar form by using the properties of isotropic tensors

$$C(k;t,t') = \theta(t-s)R(k;t,s)C(k;s,t'). \tag{41}$$

The transitivity of $\hat{R}_{\alpha r}(\mathbf{k};t,s)$ with respect to intermediate time can be proved by applying Eq. (37) to the right-hand side of itself,

$$u_\alpha(\mathbf{k},t) = \hat{R}_{\alpha\sigma}(\mathbf{k};t,s)\hat{R}_{\sigma\rho}(\mathbf{k};s,t')u_\rho(\mathbf{k},t'), \tag{42}$$

and realizing that we could also have written this as

$$u_\alpha(\mathbf{k},t) = \hat{R}_{\alpha\rho}(\mathbf{k};t,t')u_\rho(\mathbf{k},t'), \tag{43}$$

implying the result

$$\hat{R}_{\alpha\rho}(\mathbf{k};t,t') = \hat{R}_{\alpha\sigma}(\mathbf{k};t,s)\hat{R}_{\sigma\rho}(\mathbf{k};s,t') \tag{44}$$

and $t>s>t'$.

B. The mean-field propagator

The simple property of the propagator

$$R(k;t,t) = 1 \tag{45}$$

can be easily shown to be necessary by setting $s=t$ in Eq. (30). Other properties can be obtained by equating the right hand side of Eq. (30) with the right hand side of Eq. (31):

$$\theta(t-t')R(k;t,t')C(k;t',t') = \theta(t-t')\theta(t-s)R(k;t,s)C(k;s,t'). \tag{46}$$

Expanding the right hand side of Eq. (46) using Eq. (34) we obtain

$$\begin{aligned}\theta(t-t')R(k;t,t')C(k;t',t') &= [\theta(t-t')\theta(t-s)R(k;t,s) \\ &\quad \times \theta(s-t')R(k;s,t')C(k;t',t')]a \\ &\quad + [\theta(t-t')\theta(t-s)R(k;t,s) \\ &\quad \times \theta(t'-s)R(k;t',s)C(k;s,s)]b \\ &\quad - [\theta(t-t')\theta(t-s)R(k;t,s)\times\delta_{t',s}C(k;s,t')]c.\end{aligned} \tag{47}$$

Dividing the right hand side into three groups of terms labeled, respectively, a,b, and c, we will now look at Eq. (47) for two separate cases: case 1, $t>s>t'$, and case 2 $t\geqslant t'>s$.

1. Transitivity with respect to intermediate times

Here we consider case 1 corresponding to $t>s>t'$ and implying that $b=0$ and $c=0$ in Eq. (47). This will leave

$$\theta(t-t')R(k;t,t')C(k;t',t') = [\theta(t-t')\theta(t-s)R(k;t,s) \times \theta(s-t')R(k;s,t')C(k;t',t')]. \tag{48}$$

We now use the contraction property of the Heaviside function

$$\theta(t-s)\theta(s-t') = \theta(t-t') \tag{49}$$

to write Eq. (48) as

$$\theta(t-t')\underline{R(k;t,t')}C(k;t',t') = \theta(t-t')\underline{R(k;t,s)R(k;s,t')}C(k;t',t'). \tag{50}$$

From this above result, we can deduce the *transitive* property of the propagator

$$R(k;t,t') = R(k;t,s)R(k;s,t'). \tag{51}$$

This result also tells us that the transitivity of the propagator holds only for times s which are *intermediate* between the two times t and t'. This makes sense because otherwise, if s was outside the range between t and t', we would have propagation backwards in time which violates causality. This is a result which was previously only assumed [14,28] on the basis that it could be expected to follow from the corresponding relationship for the velocity-field propagator, and is now proved.

2. Linked single-time covariances

Next we consider case 2 $t\geqslant t'>s$, which corresponds to $a=0$ and $c=0$, leaving

$$\theta(t-t')R(k;t,t')\underline{C(k;t',t')} = [\theta(t-t')\theta(t-s)R(k;t,s) \times \theta(t'-s)R(k;t',s)\underline{C(k;s,s)}]. \tag{52}$$

This result is important because it links two single-time covariances. This fact becomes clearer if we take the special case of $t=t'$. This gives

$$C(k;t,t) = \theta(t-s)R(k;t,s)R(k;t,s)C(k;s,s), \tag{53}$$

implying that we need two propagators to link single-time covariances. Defining

$$\widetilde{R}(k;t,s) := R(k;t,s)R(k;t,s), \tag{54}$$

Eq. (53) can be modified to make it look like Eq. (29)

$$C(k,t) = \theta(t-s)\widetilde{R}(k;t,s)C(k,s). \tag{55}$$

Again, the presence of the unit step function ensures that the covariance can only propagate forwards in time.

III. DERIVATION OF THE LOCAL-ENERGY TRANSFER RESPONSE EQUATION

The starting point for the LET theory is the second-order renormalized covariance equation as given by Eq. (15). We can now proceed in two ways.

(1) The first is to substitute Eq. (34) in Eq. (15) and then choose $t > t'$.

(2) The second is to choose $t > t'$ and multiply both sides of Eq. (15) by $\theta(t-t')$ to show the range over which the equation will be valid. Then follow this by using the FDR, in the form of Eq. (31), throughout.

Note that it is important that we do not set $t=t'$ in the covariance equation (15) as we can only do this after evaluating the derivative of the two-time covariance.

Both methods are equivalent but the second is the easier to use in practice. Thus we begin by choosing the time ordering to be $t \geqslant t'$ and multiplying Eq. (15) by $\theta(t-t')$,

$$\theta(t-t')\frac{\partial}{\partial t}C(k;t,t') + \theta(t-t')\nu k^2 C(k;t,t') = \theta(t-t')\int d^3j\, L(\mathbf{k},\mathbf{j}) \times \left\{ \int_0^{t'} ds\, R(k;t',s)C(j;t,s)C(|\mathbf{k}-\mathbf{j}|;t,s) - \int_0^t ds\, R(j;t,s)C(k;s,t')C(|\mathbf{k}-\mathbf{j}|;t,s) \right\}. \tag{56}$$

Let us look at the first term of the left hand side of Eq. (56):

$$\begin{aligned}\theta(t-t')\frac{\partial}{\partial t}C(k;t,t') &= \frac{\partial}{\partial t}\theta(t-t')C(k;t,t') - C(k;t,t')\frac{\partial}{\partial t}\theta(t-t') \\ &= \frac{\partial}{\partial t}\theta(t-t')R(k;t,t')C(k;t',t') - C(k;t,t')\frac{\partial}{\partial t}\theta(t-t'),\end{aligned} \tag{57}$$

where we have applied the product rule in the second line, and the FDT Eq. (31) in the third line. After substituting the differential of the Heaviside unit step function

$$\frac{\partial}{\partial t}\theta(t-t') = \delta(t-t'), \tag{58}$$

we reach our final form for this part of the response equation. Thus [left hand side of Eq. (56)]

$$= \frac{\partial}{\partial t}\theta(t-t')R(k;t,t')C(k;t',t') - C(k;t,t')\delta(t-t') + \nu k^2\theta(t-t')R(k;t,t')C(k;t',t'), \tag{59}$$

where the FDR (31) was used on the second term of the left hand side of Eq. (56) also.

Now we evaluate the second time integral on the right hand side of Eq. (56), which we label as I_2:

$$I_2 = \theta(t-t')\int_0^t ds\, R(j;t,s)C(k;s,t')C(|\mathbf{k}-\mathbf{j}|;t,s). \tag{60}$$

We need to have the appropriate θ functions in front of the covariance so that the broken time-reversal symmetry becomes manifest. This information is present in the arguments of the propagator and in $\theta(t-t')$. So for $C(|\mathbf{k}-\mathbf{j}|;t,s)$

$$\theta(t-t')\int_0^t ds\, C(|\mathbf{k}-\mathbf{j}|;t,s) = \theta(t-t')\int_0^t ds\, \theta(t-s)C(|\mathbf{k}-\mathbf{j}|;t,s) \tag{61}$$

and for $C(k;s,t')$

$$\begin{aligned}\theta(t-t')\int_0^t ds\, C(k;s,t') &= \theta(t-t')\int_0^{t'} ds\, C(k;s,t') + \theta(t-t')\int_{t'}^t ds\, C(k;s,t') \\ &= \theta(t-t')\int_0^{t'} ds\, \theta(t'-s)C(k;t',s) + \theta(t-t') \times \int_{t'}^t ds\, \theta(s-t')C(k;s,t'),\end{aligned} \tag{62}$$

where we have used the property $C(k;t,t')=C(k;t',t)$ in the fourth line. With these results we can now write Eq. (60) as

$$I_2 = \left[\theta(t-t')\int_{t'}^t ds\, R(j;t,s)\theta(s-t') \times C(k;s,t')\theta(t-s)C(|\mathbf{k}-\mathbf{j}|;t,s)\right] + \left[\theta(t-t')\int_0^{t'} ds\, R(j;t,s)\theta(t'-s) \times C(k;t',s)\theta(t-s)C(|\mathbf{k}-\mathbf{j}|;t,s)\right]. \tag{63}$$

The evaluation of the first integral on the right hand side of Eq. (56) follows similarly so that the final LET response equation is given by

$$\frac{\partial}{\partial t}\theta(t-t')R(k;t,t')C(k;t',t') - C(k;t,t')\delta(t-t') + \nu k^2\theta(t-t')R(k;t,t')C(k;t',t')$$
$$= \int d^3j\, L(\mathbf{k},\mathbf{j})\,\theta(t-t')\Bigg\{\left[\int_0^{t'} ds\, R(k;t',s)\theta(t-s)\times C(j;t,s)\theta(t-s)C(|\mathbf{k}-\mathbf{j}|;t,s)\right]$$
$$-\left[\int_0^{t'} ds\, R(j;t,s)\theta(t'-s)C(k;t',s)\theta(t-s)C(|\mathbf{k}-\mathbf{j}|;t,s)\right] - \left[\int_{t'}^{t} ds\, R(j;t,s)\theta(s-t')\times C(k;s,t')\theta(t-s)C(|\mathbf{k}-\mathbf{j}|;t,s)\right]\Bigg\}. \tag{64}$$

Multiplying both sides by $\theta(t-t')$, dividing by $C(k;t',t')$, and noting that

$$\frac{\theta(t-t')C(k;t,t')}{C(k;t',t')} = \theta(t-t')R(k;t,t'), \tag{65}$$

from the FDR, in the form of Eq. (31), we reach the simplified form with the broken time-reversal symmetry manifest,

$$\theta(t-t')\left(\frac{\partial}{\partial t}+\nu k^2\right)\theta(t-t')R(k;t,t')$$
$$-\theta(t-t')R(k;t,t')\delta(t-t') + \left[\int d^3j\, L(\mathbf{k},\mathbf{j})\theta(t-t')\right.$$
$$\left.\times\int_{t'}^{t} ds\, R(j;t,s)R(k;s,t')\theta(t-s)C(|\mathbf{k}-\mathbf{j}|;t,s)\right]$$
$$= \int d^3j\, L(\mathbf{k},\mathbf{j})\theta(t-t')\int_0^{t'} ds \frac{\theta(t-s)C(|\mathbf{k}-\mathbf{j}|;t,s)}{C(k;t',t')}$$
$$\times\{R(k;t',s)\theta(t-s)C(j;t,s)$$
$$-R(j;t,s)\theta(t'-s)C(k;t',s)\}. \tag{66}$$

A. Comparison with previous forms

Apart from the addition of the second term on the left hand side

$$-\theta(t-t')R(k;t,t')\delta(t-t'), \tag{67}$$

Eq. (66) is the same as the LET response equation which appears as Eq. (3.19) in [7], Eq. (20) in [29], and Eq. (7.146) in [9]. The natural addition of this extra term as a consequence of time ordering fixes the problem of the singularity in the time derivative of the response equation (66) which occurs when one takes $t=t'$. More important, if we compare Eq. (64) with the DIA response equation (20), the additional terms on the right hand side of Eq. (64) cancel the infrared divergence and ensure compatibility with the Kolmogorov K41 spectrum.

IV. THE TWO-TIME LET THEORY

A. Partial-propagator representation

We may write the propagator in a representation which separates the discontinuous part as a Heaviside unit step function; thus

$$R(k;t,t') = \theta(t-t')\mathcal{R}(k;t,t'), \tag{68}$$

where $\mathcal{R}(k;t,t')$ is a representation of the propagator but without the discontinuity at $t=t'$. So using (68) and the FDR (27) to turn two-time covariances into single-time form, Eq. (64) for the response function becomes

$$\theta(t-t')\left(\frac{\partial}{\partial t}+\nu k^2\right)\mathcal{R}(k;t,t')$$
$$= -\theta(t-t')\int d^3j\, L(\mathbf{k},\mathbf{j})\left[\int_{t'}^{t} ds\mathcal{R}(k;s,t')\right.$$
$$\left.\times\mathcal{R}(j;t,s)\mathcal{R}(|\mathbf{k}-\mathbf{j}|;t,s)C(|\mathbf{k}-\mathbf{j}|,s)\right]$$
$$+\theta(t-t')\int d^3j\, L(\mathbf{k},\mathbf{j})\int_0^{t'} ds\left\{\mathcal{R}(k;t',s)\mathcal{R}(j;t,s)\right.$$
$$\left.\times\mathcal{R}(|\mathbf{k}-\mathbf{j}|;t,s)\frac{C(|\mathbf{k}-\mathbf{j}|,s)}{C(k,t')}[C(j,s)-C(k,s)]\right\}, \tag{69}$$

for $t\geqslant t'$. The counterterm has been canceled by use of the product rule in the time-derivative.

B. The LET closure equations

The LET equations may now be summarized as follows. For the two-time covariance, we have Eq. (15),

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)C(k;t,t')$$
$$= \int d^3j\, L(\mathbf{k},\mathbf{j})\left\{\int_0^{t'} ds\, R(k;t',s)C(j;t,s)C(|\mathbf{k}-\mathbf{j}|;t,s)\right.$$
$$\left.-\int_0^{t} ds\, R(j;t,s)C(k;s,t')C(|\mathbf{k}-\mathbf{j}|;t,s)\right\}, \tag{70}$$

and likewise Eq. (16) for the single-time covariance,

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)C(k,t) = 2\int d^3j\, L(\mathbf{k},\mathbf{j})$$
$$\times\int_0^{t} ds\, R(k;t,s)R(j;t,s)R(|\mathbf{k}-\mathbf{j}|;t,s)$$
$$\times[C(j,s)C(|\mathbf{k}-\mathbf{j}|,s)$$

$$- C(k,s)C(|\mathbf{k}-\mathbf{j}|,s)], \tag{71}$$

where we have invoked the FDR so that all two-time covariances are turned into one-time covariances.

For the response function we can use either Eq. (64) or Eq. (67). The above equations along with the generalized fluctuation-dissipation relation,

$$\theta(t-t')C(k;t,t') = \theta(t-t')R(k;t,t')C(k,t') \tag{72}$$

from which the LET is derived, and the single-time covariance link equation

$$C(k,t) = \theta(t-s)R(k;t,s)R(k;t,s)C(k,s), \tag{73}$$

complete the set of LET equations.

The LET equations have been applied, along with those of the DIA, to the problem of free decay of isotropic turbulence from arbitrary initial conditions, over a wide range of Taylor-Reynolds numbers [30,31]. In these investigations, the covariance equations on and off the time diagonal were solved simultaneously with the relevant response equation. It was later realized that for the LET theory, the response equation could be replaced by the FDR, as given by Eq. (70), and this reduced the computational effort well below that of DIA; see [7,29]. This work was for three-dimensional turbulence, while an extensive investigation of the two-dimensional case has also been carried out for DIA, SCF, and LET theories [32–34].

C. Behavior in the limit of infinite Reynolds numbers

The later two-time versions of the LET theory claimed that their solutions were compatible with K41. However, this was never shown explicitly. Compatibility with K41 is now demonstrated for the LET response or propagator equation as given by Eq. (69). We begin by writing Eq. (69) in stationary form. This means that all (single-time) covariances become time independent:

$$C(k,t) \rightarrow C(k), \tag{74}$$

and we write the propagator in relative time coordinates

$$\mathcal{R}(k;t,t') = \mathcal{R}(k;t-t'). \tag{75}$$

Next we assume the exponential form for the propagator

$$\mathcal{R}(k;t-t') = \exp[-\omega(k)(t-t')], \tag{76}$$

where, as before, $\omega(k)$ is the total eddy-decay rate. These changes result in the response equation becoming:

$$\begin{aligned}\theta(t-t')\left(\frac{\partial}{\partial t} + \nu k^2\right)\exp[-\omega(k)(t-t')] \\ = \Bigg[-\theta(t-t')\int d^3j\, L(\mathbf{k},\mathbf{j})C(|\mathbf{k}-\mathbf{j}|)\int_{t'}^{t} ds \\ \times \exp[-\omega(k)(s-t') - \omega(j)(t-s) - \omega(|\mathbf{k}-\mathbf{j}|)(t-s)]\Bigg] \\ + \Bigg[\theta(t-t')\int d^3j\, L(\mathbf{k},\mathbf{j})\int_0^{t'} ds \\ \times \Bigg\{\exp[-\omega(k)(t'-s) - \omega(j)(t-s) - \omega(|\mathbf{k}-\mathbf{j}|)(t-s)] \\ \times \frac{C(|\mathbf{k}-\mathbf{j}|)}{C(k)}[C(j)-C(k)]\Bigg\}\Bigg].\end{aligned} \tag{77}$$

Doing the differentiation, setting $t=t'$, and carrying out the time integration results in an equation for $\omega(k)$:

$$\begin{aligned}\omega(k) = \nu k^2 + \Bigg\{\int d^3j\, L(\mathbf{k},\mathbf{j})\frac{C(|\mathbf{k}-\mathbf{j}|)[C(j)-C(k)]}{C(k)[\omega(k)+\omega(j)+\omega(|\mathbf{k}-\mathbf{j}|)]} \\ \times (1-\exp\{-[\omega(k)+\omega(j)+\omega(|\mathbf{k}-\mathbf{j}|)]t\})\Bigg\},\end{aligned} \tag{78}$$

where one can ignore the last term involving the exponential as we are considering stationary systems which are time independent. Another way to justify the neglect of this term is to realize that it originates from the fact that we chose to have the initial conditions at $t=0$ rather than the more usual $t=-\infty$.

To show that Eq. (78) is not divergent we complete our analysis by substituting the inertial range and infinite Reynolds number forms for $C(k)$ and $\omega(k)$:

$$C(k) = \frac{\alpha\varepsilon^{2/3}}{4\pi}k^{-11/3}, \tag{79}$$

$$\omega(k) = \beta\varepsilon^{1/3}k^{2/3}, \tag{80}$$

where α and β are constants, and by writing the integral in k, j, μ variables

$$\begin{aligned}\omega(k) = \nu k^2 \\ + \Bigg\{\int dj \int d\mu \frac{kj^3(\mu^2-1)[\mu(k^2+j^2) - kj(1+2\mu^2)]}{k^2+j^2-2kj\mu} \\ \times \frac{\alpha\beta^{-1}\varepsilon^{1/3}|\mathbf{k}-\mathbf{j}|^{-11/3}[j^{-11/3}-k^{-11/3}]}{k^{-11/3}[k^{2/3}+j^{2/3}+|\mathbf{k}-\mathbf{j}|^{2/3}]}\Bigg\}\end{aligned} \tag{81}$$

where μ is the cosine of the angle between the two vectors $\mathbf{k}$ and $\mathbf{j}$.

There are three possible sources of divergence (of the infrared type) in this expression. However, from Eq. (81), it may be seen that the $k\rightarrow 0$ and $j\rightarrow 0$ limits do not pose a problem. The final possible source of trouble $|\mathbf{k}-\mathbf{j}|\rightarrow 0$ can be resolved by realizing that the term $[j^{-11/3}-k^{-11/3}]$ cancels the divergence caused by the $|\mathbf{k}-\mathbf{j}|^{-11/3}$ term. This is shown by expanding

$$|\mathbf{k}-\mathbf{j}|^{-11/3} = (k^2+j^2-2kj\mu)^{-11/6}, \tag{82}$$

and substituting in Eq. (81). One then Taylor expands k around j to leading order in $\epsilon=k-j$ in both the numerator and denominator of the integrand in Eq. (81). This results in the integrand becoming

$$\frac{(11/3)\alpha\beta^{-1}\varepsilon^{1/3}}{(2)^{17/6}} \times \frac{[2j^2\mu(1-\mu)-j^2](\mu^2-1)(1-\mu)^{-17/6}\epsilon}{j^{16/6}[2j^{2/3}+(2j^2)^{1/3}(1-\mu)^{1/3}]}, \tag{83}$$

and focusing on the term $(\mu^2-1)(1-\mu)^{-17/6}\epsilon$ we can see that as $\epsilon \to 0$, the integrand goes to zero, except at $\mu=1$ where the integrand is singular. This singularity can be avoided if we write the limits of the μ integral as

$$\int_{-1}^{1} d\mu \to \int_{-1}^{\uparrow 1} d\mu, \tag{84}$$

where $\uparrow 1$ implies in the limit approaching 1 from below.

This completes the analysis in the limit of infinite Reynolds number. Further information on the above technique can be found in [9].

V. SINGLE-TIME MARKOVIANIZED LET THEORY

The relevant single time LET equations are the single-time covariance (using the partial propagator form)

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)C(k,t) = 2\int d^3j\, L(\mathbf{k},\mathbf{j}) \times \int_0^t ds\, \mathcal{R}(k;t,s)\mathcal{R}(j;t,s)\mathcal{R}(|\mathbf{k}-\mathbf{j}|;t,s) \times [C(j,s)C(|\mathbf{k}-\mathbf{j}|,s) - C(k,s)C(|\mathbf{k}-\mathbf{j}|,s)], \tag{85}$$

the response equation

$$\theta(t-t')\left(\frac{\partial}{\partial t}+\nu k^2\right)\mathcal{R}(k;t,t') = -\theta(t-t')\int d^3j\, L(\mathbf{k},\mathbf{j})\left[\int_{t'}^{t} ds\, \mathcal{R}(k;s,t') \times \mathcal{R}(j;t,s)\mathcal{R}(|\mathbf{k}-\mathbf{j}|;t,s)C(|\mathbf{k}-\mathbf{j}|,s)\right] + \theta(t-t')\int d^3j\, L(\mathbf{k},\mathbf{j})\int_0^{t'} ds\left\{\mathcal{R}(k;t',s)\mathcal{R}(j;t,s) \times \mathcal{R}(|\mathbf{k}-\mathbf{j}|;t,s)\frac{C(|\mathbf{k}-\mathbf{j}|,s)}{C(k,t')}[C(j,s)-C(k,s)]\right\}, \tag{86}$$

and the single-time covariance link equation

$$C(k,t) = \theta(t-s)R(k;t,s)R(k;t,s)C(k,s). \tag{87}$$

Making a Markovian approximation we can write Eq. (85) as

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)C(k,t) = 2\int d^3j\, L(\mathbf{k},\mathbf{j})D(k,j,|\mathbf{k}-\mathbf{j}|;t) \times C(|\mathbf{k}-\mathbf{j}|,t)[C(j,t)-C(k,t)], \tag{88}$$

where the Markovian approximation amounts to updating each $C(s)$ to $C(t)$, and where

$$D(k,j,|\mathbf{k}-\mathbf{j}|;t) = \int_0^t ds\, \mathcal{R}(k;t,s)\mathcal{R}(j;t,s)\mathcal{R}(|\mathbf{k}-\mathbf{j}|;t,s) \tag{89}$$

is the *memory time*.

We now need some way of computing $D(k,j,|\mathbf{k}-\mathbf{j}|;t)$; that is, of updating it. We do this by differentiating Eq. (89) with respect to t,

$$\frac{\partial}{\partial t}D(k,j,|\mathbf{k}-\mathbf{j}|;t) = 1 + \int_0^t ds\left[\left(\frac{\partial}{\partial t}\mathcal{R}(k;t,s)\right)\mathcal{R}(j;t,s)\mathcal{R}(|\mathbf{k}-\mathbf{j}|;t,s) + \mathcal{R}(k;t,s)\left(\frac{\partial}{\partial t}\mathcal{R}(j;t,s)\right)\mathcal{R}(|\mathbf{k}-\mathbf{j}|;t,s) + \mathcal{R}(k;t,s)\mathcal{R}(j;t,s)\left(\frac{\partial}{\partial t}\mathcal{R}(|\mathbf{k}-\mathbf{j}|;t,s)\right)\right]. \tag{90}$$

To evaluate Eq. (90) we need to know the dynamical behavior of $\mathcal{R}(k;t,s)$. We obtain this from Eq. (86). Proceed by writing Eq. (86) in Langevin form

$$\theta(t-t')\left[\frac{\partial}{\partial t}+\nu k^2+\eta(k;t,t')\right]\mathcal{R}(k;t,t') = 0, \tag{91}$$

where

$$\eta(k;t,t') = \theta(t-t')\int d^3j\, L(\mathbf{k},\mathbf{j})\left[\int_{t'}^{t} ds\, \mathcal{R}(k;s,t') \times \mathcal{R}(j;t,s)\mathcal{R}(|\mathbf{k}-\mathbf{j}|;t,s)C(|\mathbf{k}-\mathbf{j}|,s)\right] - \theta(t-t')\int d^3j\, L(\mathbf{k},\mathbf{j})\int_0^{t'} ds\left\{\mathcal{R}(k;t',s)\mathcal{R}(j;t,s) \times \mathcal{R}(|\mathbf{k}-\mathbf{j}|;t,s)\frac{C(|\mathbf{k}-\mathbf{j}|,s)}{C(k,t')}[C(j,s)-C(k,s)]\right\} \tag{92}$$

is the turbulent eddy-decay rate and is obtained by comparison with Eq. (86).

Rearranging Eq. (91) we obtain

$$\theta(t-t')\frac{\partial}{\partial t}\mathcal{R}(k;t,t') = -\theta(t-t')[\nu k^2+\eta(k;t,t')]\mathcal{R}(k;t,t'), \tag{93}$$

and this allows us to write Eq. (90) as

$$\frac{\partial}{\partial t}D(k,j,|\mathbf{k}-\mathbf{j}|;t) = 1 - \int_0^t ds\{\mathcal{R}(k;t,s)\mathcal{R}(j;t,s)\mathcal{R}(|\mathbf{k}-\mathbf{j}|;t,s) \times [(\nu k^2+\nu j^2+\nu|\mathbf{k}-\mathbf{j}|^2)+\eta(k;t,s) + \eta(j;t,s)+\eta(|\mathbf{k}-\mathbf{j}|;t,s)]\}. \tag{94}$$

To be able to calculate Eq. (94) we need to take the Markovian step

$$\eta(k;t,s) \to \eta(k,t). \tag{95}$$

We can justify this step by looking at Eqs. (87), (88), and (91). Equation (91) has a general solution

$$\mathcal{R}(k;t,t') = \exp\left\{-\nu k^2(t-t') - \int_{t'}^{t} ds\ \eta(k;s,t')\right\}. \tag{96}$$

If we write Eq. (88) in the suggestive form

$$\left[\frac{\partial}{\partial t} + 2\nu k^2 + 2\xi(k,t)\right]C(k,t) = 0, \tag{97}$$

where

$$\xi(k,t) = -\int d^3j\ L(\mathbf{k},\mathbf{j})D(k,j,|\mathbf{k}-\mathbf{j}|;t) \times \frac{C(|\mathbf{k}-\mathbf{j}|,t)}{C(k,t)}[C(j,t)-C(k,t)], \tag{98}$$

then we can write the general solution of Eq. (97) as

$$C(k,t) = \exp\left\{-2\nu k^2(t-t') - 2\int_{t'}^{t} ds\ \xi(k,s)\right\}C(k,t') = [\exp\left\{-\nu k^2(t-t') - \int_{t'}^{t} ds\ \xi(k,s)\right\}]^2 C(k,t'). \tag{99}$$

Writing Eq. (87) as

$$C(k,t) = \theta(t-t')\mathcal{R}(k;t,t')\mathcal{R}(k;t,t')C(k,t') \tag{100}$$

and comparing with Eq. (99), this suggests

$$\mathcal{R}(k;t,t') = \exp\left\{-\nu k^2(t-t') - \int_{t'}^{t} ds\ \xi(k,s)\right\}. \tag{101}$$

But comparing this with Eq. (96), we see that

$$\int_{t'}^{t} ds\ \xi(k,s) = \int_{t'}^{t} ds\ \eta(k;s,t'). \tag{102}$$

Comparing the forms of $\xi(k,s)$ and $\eta(k;t,s)$, Eqs. (92) and (98), we find

$$\xi(k,s) = \eta(k;s,s). \tag{103}$$

Also in Eq. (102), since both t and t' are arbitrary, such that we can make $t \sim t'$, we may make the important assumption that

$$\eta(k;s,t') = \xi(k,s) = \eta(k;s,s) = \eta(k,s). \tag{104}$$

This tells us that in the case of the $\eta(k;s,t')$ term, we need only concern ourselves with the on-diagonal terms[1] $\eta(k;s,s) = \eta(k,s)$, which is a Markovian simplification.

[1]Leslie [5] in deriving an equation for $\eta(k,t)$ from the DIA, averages over the second time argument, i.e., $\eta(k,t) = \int_0^t ds\ \eta(k;t,s)$, whereas we simply take the on-diagonal terms. In effect Leslie's $\eta(k,t)$ should be written as $\bar{\eta}(k,t)$ showing that it is an averaged quantity.

Going back to Eq. (94), we can now write it as

$$\frac{\partial}{\partial t}D(k,j,|\mathbf{k}-\mathbf{j}|;t) = 1 - [(\nu k^2 + \nu j^2 + \nu|\mathbf{k}-\mathbf{j}|^2) + \eta(k,t) + \eta(j,t) + \eta(|\mathbf{k}-\mathbf{j}|,t)]D(k,j,|\mathbf{k}-\mathbf{j}|;t), \tag{105}$$

which along with Eq. (88) can be used to evolve the memory time.

Single-time Markovianized LET equations

The final equations for the single-time evolution may now be summarized as

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right)C(k,t) = 2\int d^3j\ L(\mathbf{k},\mathbf{j}) \times D(k,j,|\mathbf{k}-\mathbf{j}|;t) \times C(|\mathbf{k}-\mathbf{j}|,t)[C(j,t)-C(k,t)] = -2\eta(k,t)C(k,t), \tag{106}$$

$$\eta(k,t) = -\int d^3j\ L(\mathbf{k},\mathbf{j})D(k,j,|\mathbf{k}-\mathbf{j}|;t) \times \frac{C(|\mathbf{k}-\mathbf{j}|,t)}{C(k,t)}[C(j,t)-C(k,t)] \tag{107}$$

and

$$\frac{\partial}{\partial t}D(k,j,|\mathbf{k}-\mathbf{j}|;t) = 1 - [(\nu k^2 + \nu j^2 + \nu|\mathbf{k}-\mathbf{j}|^2) + \eta(k,t) + \eta(j,t) + \eta(|\mathbf{k}-\mathbf{j}|,t)]D(k,j,|\mathbf{k}-\mathbf{j}|;t). \tag{108}$$

These equations can be solved numerically with some suitable choice of initial conditions:

$$C(k,t=0) = \frac{E(k,t=0)}{4\pi k^2}, \tag{109}$$

where $E(k,t=0)$ is an arbitrarily chosen initial energy spectrum, and

$$D(k,j,|\mathbf{k}-\mathbf{j}|;t=0) = 0. \tag{110}$$

The last of these initial conditions follows from the definition of $D(k,j,|\mathbf{k}-\mathbf{j}|;t)$ in Eq. (87) and this in turn implies, from Eq. (107), that $\eta(k,t=0)=0$, as is expected, because the cascade has not yet begun at $t=0$.

This set of equations is almost identical to those of the test field model (TFM) [35], the exception being an extra term on the right hand side of (107) when compared with the corresponding TFM equation. As before, this extra term guarantees compatibility with K41. However, it remains to be seen how well the single-time LET theory performs when computed for the standard test problems.

VI. CONCLUSION

We have seen that our time-ordering procedures, as reported in [1] have allowed us to tidy up some aspects of the

LET theory. In particular, we have been able to derive a single-time form of the theory, which we have Markovianized so that it can be compared with well-known models such as the TFM or EDQNM. Such a comparison will require numerical computation and this will be the subject of further work. However, we shall conclude here with some remarks about the role of the fluctuation-dissipation relation in the Euleran two-time closures, DIA, SCF, and LET theories and then give a brief overview of the subject of spectral closures.

As we have seen, LET uses the FDR either to derive the response equation or to be used instead of a response equation. That is, with the second-order covariance equations for $C(k;t,t')$ and $C(k;t,t)$ we can specify $R(k;t,t')$ through the FDR and this gives us the requisite set of three equations.

In contrast, SCF theory works with Eq. (16) for $C(k;t,t)$, the DIA response equation (20) for $R(k;t,t')$, and the FDR to calculate $C(k;t,t')$. Calculations based on these three equations are known to agree quite closely with those for the DIA, consisting of Eqs. (15), (16), and (20).

In the case of the DIA, one may test the idea of a FDR by introducing a modified response function $R'(k;t,t')$, such that

$$R'(k;t,t') = \frac{Q(k;t,t')}{Q(k;t',t')}. \tag{111}$$

This quantity plays no part in the calculation. However, at each stage, R' can be calculated from the above relationship and compared with the actual DIA response function R at the same stage. It is this comparison that is the basis of the observation that the FDR is quite a good approximation at smaller wave numbers but is less good in the dissipation range [20]. However, such a comparison assumes that the DIA response equation is "right" and the FDR is "wrong". In fact we know that the DIA response equation does not possess the correct behaviour at large Reynolds numbers and therefore cannot be a standard of comparison. In our view, the comparison of DIA with LET is a fairer test of the use of the FDR for turbulence.

Our derivation of the FDR [1] is correct to second order in renormalized perturbation theory. Accordingly it is an approximation, but no more an approximation than the second-order covariance equations (15) and (16). Therefore its use with these equations, as in the LET theory, is entirely consistent. Nevertheless, we should draw a distinction between this situation and that in microscopic equilibrium systems, where the linear form (27) holds to all orders in perturbation theory.

An interesting feature of the present, nonequilibrium situation is that the relationship is local in wave number, despite the fact that in principle all Fourier modes are coupled by the Navier-Stokes equation. As we have indicated earlier, this has its roots in the assumption that the response of the system may be determined from an energy balance which is local in wave number. Originally this particular approach started out from a consideration of the energy flux (or transport power) through some particular mode [12] and is a weaker assumption of localness than later became of interest in the context of triad interactions (e.g., see [36–38]), and of course the theory still contains integrals over all wave numbers and intermediate times. This structure is characteristic of renormalized theories and implies the existence of effective short-range interactions. It would be interesting to see if it can be understood in terms of the later studies of localness of triad interactions and this will be the subject of further work.

Last, in order to put our results in perspective, we shall give a brief overview of the subject of spectral closures, and expand on some points as promised earlier. In particular there is another point of view about the failure of the pioneering Eulerian closures to be compatible with the Kolmogorov spectrum.

In this paper we have taken the view that this failure can be seen as an infrared divergence in the response equation in the limit of infinite Reynolds numbers; at least in the time-independent formulation [39]. This may seem rather a severe test but in reality it is just a way of diagnosing lack of scale invariance in a theory. In contrast, Kraichnan, dealing with the more complicated two-time formulation, attributed the problem to a failure to distinguish in an Eulerian frame between convective and inertial transfer effects. In order to circumvent this problem he introduced a Lagrangian-history formulation of the DIA, based on a velocity field $\mathbf{u}(\mathbf{x},t|s)$ which is defined to be the velocity at time s of a fluid particle which was at $\mathbf{x}$ at time t. This gives a mixture of Lagrangian (s is the measuring time) and Eulerian (t is the labeling time) characteristics and with four-time rather than two-time correlation and response functions, the theory results in a set of equations which are rather more complicated than those of Eulerian DIA [40].

In order to produce a tractable set of equations, Kraichnan made an abridgement which is based on setting $t'=t$ and $s'=t$, leaving correlation and response functions with a mixed two-time dependence on t and s (see [41] for a more recent justification). Later, simpler versions of Lagrangian-history theories were produced, purely in terms of measuring times [42,43], and these seem closer to Eulerian forms, in particular to the two-time form of the SCF theory [21].

It is our intention to return to these matters in more detail when we are in a position to present calculations of the single-time LET equations on standard test problems. We believe that a significant result of our current line of approach can be to help clarify issues between the various theoretical approaches, some of which have been outstanding for many years. However we should end with a caveat. All the work presented here belongs to a class of theories based on the second-order truncation of the renormalized perturbation expansion of the Navier-Stokes equation. It should be emphasized therefore, that, although these theories have a general character, nevertheless they have no systematic means of controlling or even estimating their own errors. Thus although they seem to represent the best that we can do at the moment, this caveat must be borne firmly in mind.

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